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The Gradient Condition for One-Dimensional Symmetric Exclusion Processes

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For every Gibbs measure on the one dimensional lattice **Z** with translation-invariant potential of finite range, an exchange rate for one-dimensional lattice gas which satisfy both the detailed balance condition relative to the Gibbs measure and the gradient condition is constructed.

KEY WORDS: Gibbs measure; exclusion; detailed balance condition; gradient condition.

1. INTRODUCTION

Given a Gibbs measure on the one dimensional lattice **Z** with translation-invariant potential of finite range, we construct an exchange rate for one-dimensional lattice gas which satisfy both the detailed balance condition relative to the Gibbs measure and the gradient condition. For the construction, we use an infinite system of linear equations indexed by finite sets which is given in ref. 4. Since this system of equations has plenty of freedom, it has many solutions, most of which do not possess properties necessary for constructing the desired exchange rate. Our strategy is to find a suitable condition such that the system with it added becomes uniquely solvable and the unique solution satisfies the required properties.

Based on an exchange rate which satisfies both the detailed balance condition and the gradient condition, we can prove the hydrodynamic limit for every one-dimensional lattice gas reversible under the Gibbs measure that is not necessarily of gradient type, in a way parallel to refs. 2 and 5 with the help of the result of ref. 3 on the spectral gap.

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It is sometimes interesting to study an asymmetric process which is obtained by modifying a symmetric exchange rate (i.e., one satisfying the detailed balance condition) through a bias, while the class of invariant measures for biased process is not known in general. But if the symmetric exchange rate satisfies the gradient condition, then the class of invariant measures for the biased process coincides with that for the original symmetric process. Moreover, the converse is shown to be true, namely, such coincidence of the classes of the invariant measures implies the gradient condition of the symmetric process. (See Section 5 for details.)

Let $\eta = (\eta_x; x \in \mathbf{Z})$, $\eta_x = 0$ or 1, denotes an element of $\{0, 1\}^{\mathbf{Z}}$, the state space of one-dimensional lattice gas. The site x is interpreted as vacant if $\eta_x = 0$ and occupied if $\eta_x = 1$. The potential $\{J_A\}_{A \subset \mathbf{Z}}$ is supposed to have a finite range: there exists a constant p such that

$$J_A = 0 \qquad \text{whenever} \quad \operatorname{diam} A > p \tag{1}$$

and to be translation-invariant:

$$J_A = J_{A+a} \qquad \text{for all} \quad A \subset \mathbf{Z} \text{ and } a \in \mathbf{Z} \tag{2}$$

We define a Hamiltonian $H_\Lambda(\eta) = H^J_\Lambda(\eta)$ by

$$H_\Lambda(\eta) := \sum_{A \subset \mathbf{Z},\, A \cap \Lambda \neq \varnothing} J_A \eta^A, \qquad \eta^A := \prod_{x \in A} \eta_x$$

and a shift operator τ_x by

$$\begin{aligned} (\tau_x \eta)_z &= \eta_{x+z} && \text{for all} \quad x, z \in \mathbf{Z}, \\ \tau_x A &= A + x && \text{for all} \quad A \subset \mathbf{Z} \text{ and } x \in \mathbf{Z} \end{aligned}$$

Our main result is stated as follows.

Theorem 1.1. There exists an exchange rate $c(x, x+1, \eta)$ which satisfies the following conditions:

1. Locality: $c(x, x+1, \eta)$ depends only on $\{\eta_z; |z - x| \leqslant r\}$ for some $r > 0$.
2. Translation invariance: $c(x, x+1, \eta) = c(0, 1, \tau_x \eta)$, for all $x \in \mathbf{Z}$.
3. Positivity and exclusion: $c(x, x+1, \eta) > 0$ if $\eta_x \neq \eta_{x+1}$, and $c(x, x+1, \eta) = 0$ if $\eta_x = \eta_{x+1}$.
4. Detailed balance condition:

$$\begin{aligned} &c(x, x+1, \eta) \exp[-H_{\{0,1\}}(\eta)] \\ &\quad = c(x, x+1, \eta^{x,x+1}) \exp[-H_{\{0,1\}}(\eta^{x,x+1})] \end{aligned} \tag{3}$$

where

$$(\eta^{x,y})_z = \begin{cases} \eta_y & \text{if} \quad z = x, \\ \eta_x & \text{if} \quad z = y, \\ \eta_z & \text{otherwise} \end{cases}$$

5. Gradient condition: there exists a local function $h(\eta)$ such that

$$c(x, x+1, \eta)(\eta_x - \eta_{x+1}) = h(\tau_x \eta) - h(\tau_{x+1} \eta) \tag{4}$$

Remark 1.2. Uniqueness of the exchange rate satisfying the conditions given in Theorem 1.1 does not hold. Presumably there exists a family of such exchange rates that involves continuum of parameters for each potential. We have no proof in general, but there are examples in refs. 1 and 4.

Remark 1.3. The function $c(x, x+1, \eta)$ that we shall construct depends only on $\{\eta_z; z \in \{x-p, x-p+1, ..., x+p, x+p+1\}\}$; Hence the function $h(\eta)$ of (4) depends only on $\{\eta_z; z \in \{-p, -p+1, ..., p-1, p\}\}$.

2. A SYSTEM OF LINEAR EQUATIONS FOR $\{a(A)\}$

In this section we describe a system of linear algebraic equations given in ref. 4. By the condition 2 we have only to consider the case $x = 0$. We define $\Delta H(\eta)$ by

$$\Delta H(\eta) = \sum_A^* (J_{A \cup \{0\}} - J_{A \cup \{1\}}) \eta^A \tag{5}$$

Here $\sum_A^*$ stands for summation over all finite subsets $A \subset \mathbf{Z}$ which contain neither 0 nor 1. Note that $\Delta H(\eta)$ does not depend on (η_0, η_1). By the conditions 3 and 4 of Theorem 1.1 $c(0, 1, \eta)$ must be given in the form,

$$c(0, 1, \eta) = \eta_0(1 - \eta_1) g(\eta) + (1 - \eta_0) \eta_1 g(\eta) e^{-\Delta H(\eta)} \tag{6}$$

where g is a positive local function which dose not depend on η_0, η_1. Let us write $A \subset\subset \mathbf{Z}$ if A is a finite subset of $\mathbf{Z}$ and if we expand the function h appearing in the gradient condition in the form

$$h(\eta) = \sum_{A \subset\subset \mathbf{Z}} a(A) \eta^A \tag{7}$$

Then

$$h(\tau_1\eta) - h(\eta) = \sum_{A \subset\subset \mathbf{Z}} \{a(\tau_{-1}A) - a(A)\}\, \eta^A \tag{8}$$

We rewrite (6) as

$$c(0, 1, \eta)(\eta_1 - \eta_0) = -\eta_0 g(\eta) + \eta_1 g(\eta)\, e^{-\Delta H(\eta)} + \eta_0\eta_1 g(\eta)(1 - e^{-\Delta H(\eta)})$$

Equating the right side of this with that of (8) and comparing the coefficient of $1, \eta_0, \eta_1$ and $\eta_0\eta_1$ we deduce the following system of equations

$$0 = \sum_A{}^* \{a(\tau_{-1}A) - a(A)\}\, \eta^A \tag{9}$$

$$-g(\eta) = \sum_A{}^* \{a(\tau_{-1}(A \cup \{0\})) - a(A \cup \{0\})\}\, \eta^A \tag{10}$$

$$g(\eta)\, e^{-\Delta H(\eta)} = \sum_A{}^* \{a(\tau_{-1}(A \cup \{1\})) - a(A \cup \{1\})\}\, \eta^A \tag{11}$$

$$g(\eta)(1 - e^{-\Delta H(\eta)}) = \sum_A{}^* \{a(\tau_{-1}(A \cup \{0, 1\})) - a(A \cup \{0, 1\})\}\, \eta^A \tag{12}$$

The equation (9) holds if and only if

$$a(\tau_{-1}A) = a(A) \qquad \text{for all} \quad A \subset\subset \mathbf{Z}\backslash\{0, 1\} \tag{13}$$

Since the sum on the left sides of the equations (10)–(12) vanishes, they imply that

$$\begin{aligned} &a(\tau_{-1}(A \cup \{0\})) - a(A \cup \{0\}) + a(\tau_{-1}(A \cup \{1\})) - a(A \cup \{1\}) \\ &\quad + a(\tau_{-1}(A \cup \{0, 1\})) - a(A \cup \{0, 1\})\} \\ &= 0, \qquad \text{for all} \quad A \subset\subset \mathbf{Z}\backslash\{0, 1\} \end{aligned} \tag{14}$$

Since $e^{\Delta H(\eta)}$ dose not depend on η_0, η_1 we can expand it in the form

$$e^{\Delta H(\eta)} = \sum_B{}^* d(B)\, \eta^B \tag{15}$$

From the equations (10), (11) and (15) it then follows that

$$\begin{aligned} &a(\tau_{-1}(A \cup \{0\})) - a(A \cup \{0\}) \\ &\quad + \sum_{B \cup C = A} \{a(\tau_{-1}(C \cup \{1\})) - a(C \cup \{1\})\}\, d(B) \\ &= 0, \qquad \text{for all} \quad A \subset\subset \mathbf{Z}\backslash\{0, 1\} \end{aligned} \tag{16}$$

Conversely, if a collection $\{a(A)\}_{A \subset\subset \mathbf{Z}}$ solves (13), (14) and (16), and we define $g(\eta)$ by (10), then we have the equations (9) through (12), so that the exchange rate given by (6) satisfies both the gradient condition and the detailed balance condition, provided that the function $h(\eta)$ given by (7) is local and the function $g(\eta)$ given by (10) satisfies $g(\eta) > 0$.

3. NOTATIONS AND SOME RESUTLS

For $A \subset \mathbf{Z}\backslash\{0, 1\}$ we define $\Delta H(A)$ by

$$\Delta H(A) = \sum_{B \subset\subset A} (J_{B \cup \{0\}} - J_{B \cup \{1\}})$$

It immediately follows that

$$\Delta H(\eta) = \Delta H(S(\eta)\backslash\{0, 1\})$$

where $S(\eta)$ is the support of η, i.e., $S(\eta) = \{x \in \mathbf{Z} : \eta_x = 1\}$.

We consider the sets

$$\Lambda = \{-p, -p+1, ...\} \qquad \text{and} \qquad \Gamma = \{-p, -p+1, ..., p, p+1\}$$

In view of (1) it holds that

$$\Delta H(A) = \Delta H(A \cap \Gamma) \tag{17}$$

$$\Delta H(\varnothing) = 0 \tag{18}$$

Remark 3.1. In the sequel we shall make use of some elementary formulas on the summation over subsets of a finite set, which are recalled here.

(i) By the binomial expansion of $(1-1)^{\#A}$ we have $\sum_{B \subset A} (-1)^{\#(A\backslash B)} = 0$ if $A \neq \varnothing$.

(ii) Let Ω be a finite set. If f and g are functions of subsets of Ω, then by using (i) it is easy to check that the following two conditions are equivalent.

1. $f(A) = \sum_{B \subset A} g(B)$ for all $A \subset \Omega$.
2. $g(A) = \sum_{B \subset A} (-1)^{\#(A\backslash B)} f(B)$ for all $A \subset \Omega$.

Lemma 3.2. The coefficients $d(B)$ defined by (15) satisfies that

$$d(B) = 0 \qquad \text{for all} \quad B \subset \mathbf{Z}\backslash\{0, 1\} \text{ such that } B \cap \Gamma^c \neq \varnothing$$

Proof. Decompose B into $C = B \cap \Gamma$ and $D = B \backslash \Gamma$. Then by Remark 3.1 and (17)

$$\begin{aligned} d(B) &= \sum_{E \subset B} (-1)^{\#(B \backslash E)} e^{\Delta H(E)} \\ &= \sum_{G \subset D} (-1)^{\#(D \backslash G)} \sum_{F \subset C} (-1)^{\#(C \backslash F)} e^{\Delta H(F)} \end{aligned}$$

but the first factor of the last line equals zero according to Remark 3.1(i). ∎

Lemma 3.3. If a collection $\{a(A)\}_{A \subset\subset \mathbf{Z}}$ satisfies the following two conditions

(i) $a(A) = 0$ for all A such that $A \cap \Lambda^c \neq \emptyset$, and

(ii) $\{a(A)\}_{A \subset\subset \Lambda}$ solves the equations (13), (14), (16) for all $A \subset\subset \Lambda \backslash \{0, 1\}$,

then the collection $\{a(A)\}_{A \subset\subset \mathbf{Z}}$ solves the equations (13), (14), (16) for all $A \subset\subset \mathbf{Z} \backslash \{0, 1\}$.

Proof. We have only to check (16) for A such that $A \cap \Lambda^c \neq \emptyset$. The left side of (16) is written as

$$\begin{aligned} &a(\tau_{-1}(A \cup \{0\})) - a(A \cup \{0\}) \\ &\quad + \sum_{D \subset A,\, D \cap \Lambda^c \neq \emptyset,\, D \cup E = A} \{a(\tau_{-1}(E \cup \{1\})) - a(E \cup \{1\})\}\, d(D) \\ &\quad + \sum_{D \subset (A \cap \Lambda),\, D \cup E = A} \{a(\tau_{-1}(E \cup \{1\})) - a(E \cup \{1\})\}\, d(D) \end{aligned}$$

of which the second term vanishes, since if $D \subset A$ and $D \cap \Lambda^c \neq \emptyset$, then $d(D) = 0$ by Lemma 3.2. By condition (i), $a(A \cup \{0\}) = a(\tau_{-1}(A \cup \{0\})) = 0$, and if $E \subset A$ and $E \cap \Lambda^c \neq \emptyset$ then $a(E \cup \{1\}) = a(\tau_{-1}(E \cup \{1\})) = 0$, so that the other two terms also vanish. ∎

Put

$$\tilde{a}(A) = a(A) - a(\tau_{-1} A) \tag{19}$$

Then it is easy to check that the conditions (i) and (ii) of Lemma 3.3 hold if and only if the following system of equations holds

$$a(A)=0 \qquad \text{whenever} \quad A \cap \Lambda^c \neq \varnothing \tag{20}$$

$$\tilde{a}(A)=0 \qquad \text{for all} \quad A \subset\subset \Lambda\backslash\{0,1\} \tag{21}$$

$$\sum_{D \subset A} \{\tilde{a}(D \cup \{0\}) + \tilde{a}(D \cup \{1\}) + \tilde{a}(D \cup \{0,1\})\} = 0$$
$$\text{for all} \quad A \subset\subset \Lambda\backslash\{0,1\} \tag{22}$$

$$\sum_{D \subset A} \tilde{a}(D \cup \{0\}) + \sum_{D \subset A} \tilde{a}(D \cup \{1\})\, e^{\Delta H(A)} = 0$$
$$\text{for all} \quad A \subset\subset \Lambda\backslash\{0,1\} \tag{23}$$

Recall the remark given at the end of Section 2, where we state the conditions for $h(\eta)$ and $g(\eta)$, that is, $h(\eta)$ is local and $g(\eta)>0$. These are written as

$$\sum_{D \subset A} \tilde{a}(D \cup \{0\}) > 0 \qquad \text{for all} \quad A \subset\subset \Lambda\backslash\{0,1\} \tag{24}$$

and

$$a(A)=0 \qquad \text{whenever} \quad A \cap C^c \neq \varnothing \tag{25}$$

where C is some finite set.

Given a set function $\{b(A)\}_{A \subset\subset \Lambda\backslash\{0,1\}}$, we introduce an additional system of equations

$$b(A) = \sum_{D \subset A} \tilde{a}(D \cup \{0\}) \tag{26}$$

so that we will get a unique solution of (19)–(23) and (26). If $b(A)>0$ for all $A \subset\subset \Lambda\backslash\{0,1\}$, then the unique solution satisfies (24). Thus our problem of constructing h is solved if we can find $b(A)>0$ so that the corresponding solution $a(A)$ also satisfies (25). Our proof of Theorem 1.1 in the next section consists of proving (25) for a suitably chosen $\{b(A)\}$.

4. CONSTRUCTING AN EXCHANGE RATE

Definition 4.1. We define a mapping $\tilde{\tau}$ from all finite subsets of $\mathbf{Z}\backslash\{0,1\}$ into themselves by

$$\tilde{\tau}A = \begin{cases} \tau_1 A & -1 \notin A, \\ (\tau_2 A\backslash\{1\}) \cup \{2\} & -2 \notin A, \ -1 \in A, \\ \vdots & \\ (\tau_k A\backslash\{1\}) \cup \{k\} & -k \notin A, \ \{-1, -2,..., -(k-1)\} \subset A, \\ \vdots & \end{cases}$$

and then $\tilde{\tau}^n$, $n \geqslant 2$, inductively by $\tilde{\tau}^n A = \tilde{\tau}(\tilde{\tau}^{n-1} A)$. We define $\tilde{\tau}^{-1}$ in the same way but with the position $-k$ replaced by $k+1$ and at the same time the shift τ_k by τ_{-k}, and define $\tilde{\tau}^n$ for $n < -1$ by iteration. Clearly $\tilde{\tau}^{-1}$ is the inverse of $\tilde{\tau}$.

For $A \subset\subset \mathbf{Z} \backslash \{0, 1\}$ we define $b(A)$ by

$$b(A) = \prod_{n=1}^{\infty} e^{-\Delta H(\tilde{\tau}^n A)} \tag{27}$$

From (17) and (18) it follows that

$$b(A) = \prod_{n=1}^{|\min A| + p} e^{-\Delta H(\tilde{\tau}^n A)} \qquad \text{for all} \quad A \subset\subset \mathbf{Z} \backslash \{0, 1\}$$

Lemma 4.2. $\{b(A)\}$ has the following properties

1. Locality:

$$b(A) = b(A \cap \Gamma) \qquad \text{for all} \quad A \subset\subset \mathbf{Z} \backslash \{0, 1\} \tag{28}$$

2. Relation between $b(A)$ and $b(\tilde{\tau} A)$:

$$b(A) = b(\tilde{\tau} A)\, e^{-\Delta H(\tilde{\tau} A)} \qquad \text{for all} \quad A \subset\subset \mathbf{Z} \backslash \{0, 1\} \tag{29}$$

3. Positivity:

$$b(A) > 0 \qquad \text{for all} \quad A \subset\subset \mathbf{Z} \backslash \{0, 1\} \tag{30}$$

The property 1 will follow from the next lemma.

Lemma 4.3. It holds that

$$\sum_{k=-\infty}^{\infty} \Delta H(\tilde{\tau}^k A) = 0 \qquad \text{for all} \quad A \subset\subset \mathbf{Z} \backslash \{0, 1\} \tag{31}$$

Proof. Let $A \subset\subset \mathbf{Z} \backslash \{0, 1\}$ and k be the number of connected components of A. We may suppose $\min A = 2$ since there can always be found n such that $\min \tilde{\tau}^n A = 2$. We write

$$\tilde{A}_1 = A = A_1 \cup A_2 \cup \cdots \cup A_k$$

where A_i are connected components arranged in order from the left to the write; $A_i = \{a_i, a_i+1,..., b_i\}$, $2 = a_1 \leqslant b_1 < a_2 - 1 < b_2 < \cdots < a_k - 1 < b_k$. We define $\tilde{A}_l$ $(1 \leqslant l \leqslant 2k)$ by

$$\tilde{A}_{2i-1} = \tilde{\tau}^{f(i)} \tilde{A}_1, \qquad \tilde{A}_{2i} = \tilde{\tau}^{f(i)-1} \tilde{A}_1$$

where

$$f(i) = \begin{cases} 0 & i = 1, \\ -\sum_{j=2}^{i} (a_j - b_{j-1} - 1) & 2 \leqslant i \leqslant k \end{cases}$$

Notice that the mapping $\tilde{\tau}$ conserves the number of connected components as well as the number of elements. The function f is chosen so that the left end of the ith component of $\tilde{A}_{2i-1}$ is 2 and the right end of the ith component of $\tilde{A}_{2i}$ is -1. Now, if $\tilde{\tau}^k A = \tau_1 \tilde{\tau}^{k-1} A$ (i.e., $-1 \notin \tilde{\tau}^{k-1} A$), then $\sum_{D \subset \tilde{\tau}^k A} J_{D \cup \{1\}} = \sum_{D \subset \tilde{\tau}^{k-1} A} J_{D \cup \{0\}}$. So

$$-\sum_{l=-\infty}^{\infty} \Delta H(\tilde{\tau}^l A) = \sum_{i=1}^{k} \left(\sum_{D \subset \tilde{A}_{2i-1}} J_{D \cup \{1\}} - \sum_{D \subset \tilde{A}_{2i}} J_{D \cup \{0\}} \right) \tag{32}$$

We must show that the right side of (32) vanishes. To this end we construct the one-to-one mapping from $\bigcup_{i=1}^{k} \mathscr{P}(\tilde{A}_{2i-1})$ into $\bigcup_{i=1}^{k} \mathscr{P}(\tilde{A}_{2i})$, where $\mathscr{P}(A)$ denotes a power set of A. To define the mapping, first we decompose $\tilde{A}_i$ into connected components $A_{i,j}$ for $1 \leqslant j \leqslant k$:

$$\tilde{A}_i = A_{i,1} \cup A_{i,2} \cup \cdots \cup A_{i,k}$$

Now consider a subset $D \subset \tilde{A}_{2i-1}$. Put $D_j = D \cap A_{2i-1,j}$ if $j \neq i$, and $D_i = (D \cap A_{2i-1,i}) \cup \{1\}$, and define k_{D_j} by

$$k_{D_j} = \begin{cases} \min\{k : k > 0, \tau_k D_j \cap A^c_{2i-1,j} \neq \varnothing\} - 1 & D_j \neq \varnothing, \\ \infty & D_j = \varnothing \end{cases}$$

Because $k_{D_i} < \infty$, there exists $1 \leqslant p \leqslant k$ and positive integers $j_1, j_2,..., j_p$ which satisfy $k_{D_{j_1}} = k_{D_{j_2}} = \cdots = k_{D_{j_p}} < k_{D_m}$ for $m \neq j_q$ $(q = 1, 2,..., p)$. We order the number $1, 2,..., k$ by $i, i+1,..., k, 1, 2,..., i-1$ and let j_q be the first member of $\{j_1,..., j_p\}$ in this ordering. Then we can find $E \subset \tilde{A}_{2j_q}$ such that

$$\tau_{-l}(E \cup \{0\}) = D \cup \{1\}$$

where $l = b_{j_q} - k_{D_{j_q}}$. We can determine the inverse mapping in the same manner but with the position k replaced with $-k+1$ and the ordering is reversed. It would be clear that by means of this one-to-one correspondence the sum on the right side of (32) vanishes by cancellation. ▮

Proof of Lemma 4.2. If $A = B \cup C$ where $B = A \cap (A \backslash \Gamma)$ and $C = A \backslash B$ then

$$-\Delta H(\tilde{\tau}A) = -\Delta H(\tilde{\tau}C)$$

because $(\tilde{\tau}A) \cap \Gamma$ does not depend on the part B of A. Hence

$$b(A) = b(C)$$

On the other hand, by Lemma 4.3,

$$b(A) = \prod_{k=0}^{\infty} e^{\Delta H(\tilde{\tau}^{-k}A)}$$

which shows $b(C) = b(C \cap \Gamma)$ by the same reasoning as above. Thus $b(A) = b(C \cap \Gamma) = b(A \cap \Gamma)$. The property 1 has been verified. The properties 2 and 3 are trivial by definition. ▮

Lemma 4.4. Let $b(A)$ be given by (27). Then the unique solution of (19)–(23) and (26) satisfies the condition (25).

Proof. Clearly it suffices to prove

(i) if $0 \notin A$ then $a(A) = 0$,

and

(ii) if $\operatorname{diam} A > \operatorname{diam} \Gamma$ then $a(A) = 0$.

The proof of (i) is carried out by double induction on $\# A$ and $\max A$. Given a set A such that $0 \notin A$, we will assume that $a(B) = 0$ if either $\# B < \# A$, $0 \notin B$ or $\# B = \# A$, $\max B < \max A$, $0 \notin B$. The equation (21) and the assumption imply that

$$a(A) = \tilde{a}(A) + a(\tau_{-1}A) = 0 \qquad \text{for all} \quad A \text{ such that } 0, 1 \notin A \tag{33}$$

The equations (21) and (26) imply that

$$\begin{aligned}\sum_{D \subset A} a(D \cup \{0\}) &= \sum_{D \subset A} \tilde{a}(D \cup \{0\}) + \sum_{D \subset \tau_{-1}A} a(D \cup \{-1\}) \\ &= b(A \cap A) + \sum_{D \subset \tau_{-1}A \cup \{-1\}} a(D) - \sum_{D \subset \tau_{-1}A} a(D) \\ &\qquad \text{for all} \quad A \text{ such that } 0, 1 \notin A\end{aligned} \tag{34}$$

The equations (21)–(23) and (26) imply that

$$\sum_{D \subset A} \tilde{a}(D \cup \{1\}) = -b(A \cap \Lambda)\, e^{-\Delta H(A \cap \Lambda \backslash \{1\})}$$

$$\sum_{D \subset A} \tilde{a}(D \cup \{0, 1\}) = b(A \cap \Lambda)(e^{-\Delta H(A \cap \Lambda)} - 1)$$

$$\text{for all} \quad A \text{ such that } 0, 1 \notin A$$

Hence for all A such that $0 \notin A$ and $1 \in A$ we have

$$\begin{aligned}
&\sum_{D \subset A} a(D) \\
&\quad = \sum_{D \subset A \backslash \{1\}} \{a(D \cup \{1\}) + a(D)\} \\
&\quad = \sum_{D \subset A \backslash \{1\}} \{\tilde{a}(D \cup \{1\}) + \tilde{a}(D)\} + \sum_{D \subset \tau_{-1} A \backslash \{0\}} [a(D \cup \{0\}) + a(D)] \\
&\quad = -b(A \cap \Lambda \backslash \{1\})\, e^{-\Delta H(A \cap \Lambda \backslash \{1\})} + \sum_{D \subset \tau_{-1} A \backslash \{0\}} [a(D \cup \{0\}) + a(D)]
\end{aligned} \tag{35}$$

and

$$\begin{aligned}
&\sum_{D \subset A} a(D \cup \{0\}) \\
&\quad = \sum_{D \subset A \backslash \{1\}} \{a(D \cup \{0, 1\}) + a(D \cup \{0\})\} \\
&\quad = \sum_{D \subset A \backslash \{1\}} \{\tilde{a}(D \cup \{0, 1\}) + \tilde{a}(D \cup \{0\})\} \\
&\qquad + \sum_{D \subset \tau_{-1} A \backslash \{0\}} [a(D \cup \{-1, 0\}) + a(D \cup \{-1\})] \\
&\quad = b(A \cap \Lambda \backslash \{1\})\, e^{-\Delta H(A \cap \Lambda \backslash \{1\})} - \sum_{D \subset \tau_{-1} A \backslash \{0\}} [a(D \cup \{0\}) + a(D)] \\
&\qquad + \sum_{D \subset \tau_{-1} A \cup \{-1\} \backslash \{0\}} [a(D \cup \{0\}) + a(D)]
\end{aligned} \tag{36}$$

The two equations (35) and (36) imply that

$$\sum_{D \subset A} [a(D) + a(D \cup \{0\})] = \sum_{D \subset \tau_{-1} A \cup \{-1\} \backslash \{0\}} [a(D \cup \{0\}) + a(D)] \tag{37}$$

for all A such that $0 \notin A$, $1 \in A$. By (21) and (33), we have only to consider $A \subset\subset \Lambda$ such that $0 \notin A$ and $1 \in A$. By the assumption of induction $a(A) = \sum_{D \subset A} a(D)$ for $0 \notin A$. We may suppose that $A \subset\subset \Lambda$ is a union of $\{1, 2,..., k\}$ $(k \geqslant 1)$ and $\tilde{A}$ for which $\tilde{A} \cap \{1, 2,..., k, k+1\} = \emptyset$. Then we have by (35)

$$a(A) = \sum_{D \subset \{1, 2,..., k\} \cup \tilde{A}} a(D)$$

$$= -b((\{2, 3,..., k\} \cup \tilde{A}) \cap \Lambda)\, e^{-\Delta H((\{2, 3,..., k\} \cup \tilde{A}) \cap \Lambda}$$

$$+ \sum_{D \subset \{1, 2,..., k-1\} \cup \tau_{-1}\tilde{A}} [a(D \cup \{0\}) + a(D)]$$

By (37) the last sum equals

$$\sum_{D \subset \{1, 2,..., k-2\} \cup \{-1\} \cup \tau_{-2}\tilde{A}} [a(D \cup \{0\}) + a(D)]$$

and repeating the same procedure we arrive at

$$= \sum_{D \subset \{-1, -2,..., -(k-1)\} \cup \tau_{-k}\tilde{A}} [a(D \cup \{0\}) + a(D)]$$

$$= \sum_{D \subset \{-1, -2,..., -(k-1)\} \cup \tau_{-k}\tilde{A}} a(D \cup \{0\})$$

Therefore by (34)

$$a(A) = -b((\{2, 3,..., k\} \cup \tilde{A}) \cap \Lambda)\, e^{-\Delta H((\{2, 3,..., k\} \cup \tilde{A}) \cap \Lambda)}$$

$$+ b((\{-1, -2,..., -(k-1)\} \cup \tilde{A}_{-k} \cap \Lambda))$$

which vanishes in view of (28) and (29) since $\tilde{\tau}(\{-1, -2,..., -(k-1)\} \cup \tau_{-k}\tilde{A}) = \{1, 2,..., k\} \cup \tilde{A}$. Claim (i) has been verified.

For the proof of (ii) it suffices, by virtue of the first claim (i), to prove

$$\tilde{a}(A) = 0 \qquad \text{whenever} \quad \operatorname{diam}(A) > \operatorname{diam} \Gamma \tag{38}$$

If $0, 1 \notin A$ then (38) is trivial. Suppose $0, 1 \notin B$ and consider the cases $A = B \cup \{0\}$, $B \cup \{1\}$ or $B \cup \{0, 1\}$.

We decompose B into $C = B \cap \Gamma$ and $D = B \backslash \Gamma$ and apply Remark 3.1(ii), the defining relation (26), (28) and Remark 3.1(i) in turn to see

$$\begin{aligned}\tilde{a}(B \cup \{0\}) &= \sum_{F \subset B} (-1)^{\#(B \backslash F)} b(F) \\ &= \left(\sum_{F \subset C} (-1)^{\#(C \backslash F)} b(F) \right) \left(\sum_{G \subset D} (-1)^{\#(D \backslash G)} \right) \\ &= 0\end{aligned}$$

(notice that $D \neq \emptyset$ since $\operatorname{diam} A > \operatorname{diam} \Gamma$). We can show $\tilde{a}(B \cup \{1\}) = \tilde{a}(B \cup \{0, 1\}) = 0$ in the same way. ∎

For the proof of Remark 1.3 we prove that the exchange rate constructed by the solution of (19)–(23), (26) and (27) depends only on $\{\eta_z; z \in \{-p, -p+1, ..., p, p+1\}\}$, and the function $h(\eta)$ in (4) depends only on $\{\eta_z; z \in \{-p, -p+1, ..., p-1, p\}\}$. To this end we first notice that the exchange rate $c(0, 1, \eta)$ is rewritten as

$$\begin{aligned}c(0, 1, \eta) &= \eta_0(1-\eta_1)\, g(\eta) + (1-\eta_0)\, \eta_1 g(\eta)\, e^{-\Delta H(\eta)} \\ &= \eta_0(1-\eta_1)\, b(S(\eta) \backslash \{0, 1\}) \\ &\quad + (1-\eta_0)\, \eta_1 b(S(\eta) \backslash \{0, 1\})\, e^{-\Delta H(S(\eta) \backslash \{0, 1\})}\end{aligned}$$

where $S(\eta)$ is the support of η, i.e., $S(\eta) = \{x \in \mathbf{Z} : \eta_x = 1\}$. Since $b(A) = b(A \cap \Gamma)$ and $e^{-\Delta H(A)} = e^{-\Delta H(A \cap \Gamma)}$, $c(0, 1, \eta)$ depends only on $\{\eta_z; z \in \Gamma\} = \{\eta_z; z \in \{-p, -p+1, ..., p, p+1\}\}$; hence so does $\tau_1 h(\eta) - h(\eta) = c(0, 1, \eta) \times (\eta_1 - \eta_0)$. It would be obvious that the function $h(\eta)$ depends only on $\{\eta_z; z \in \{-p, -p+1, ..., p-1, p\}\}$.

5. BIASED EXCHANGE RATE

In this section, we consider the driven lattice gas on a discrete torus $\mathbf{T}_N = \mathbf{Z}/N\mathbf{Z}$. Assume $c(x, x+1, \eta)$ satisfies the condition 1–4 of the Theorem 1.1. Let L^N be the generator defined by

$$L^N f(\eta) = \sum_{x \in \mathbf{T}_N} c(x, x+1, \eta)(f(\eta^{x, x+1}) - f(\eta))$$

and X^N be the Markov process whose generator is L^N. A rate $c_p(x, x+1, \eta)$, defined by

$$c_p(x, x+1, \eta) = p\eta_x c(x, x+1, \eta) + (1-p)\, \eta_{x+1} c(x, x+1, \eta)$$

$0 \leqslant p \leqslant 1$, is called a biased (exchange) rate. Let L_p^N be the generator defined by

$$L_p^N f(\eta) = \sum_{x \in \mathbf{T}_N} c_p(x, x+1, \eta)(f(\eta^{x, x+1}) - f(\eta))$$

and X_p^N be the Markov process whose generator is L_p^N.

Proposition 5.1. For each sufficiently large N, the class of invariant measures for X_p^N coincides with that of X^N if and only if $c(x, x+1, \eta)$ satisfies the condition 5 of Theorem 1.1.

Proof. Assume μ_N is an invariant measure of X^N, which is a Gibbs measure on $\mathbf{T}_N$ with Hamiltonian. By a simple computation,

$$\int L_p^N f(\eta)\, \mu_N(d\eta) = \int \sum_{x \in \mathbf{T}_N} (1-2p)(\eta_x - \eta_{x+1})\, c(x, x+1, \eta)\, f(\eta)\, \mu_N(d\eta) \tag{39}$$

By the condition 5 of Theorem 1.1 the right side of (39) is equal to

$$\int \sum_{x \in \mathbf{T}_N} (1-2p)(\tau_x h(\eta) - \tau_{x+1} h(\eta))\, f(\eta)\, \mu_N(d\eta) = 0;$$

hence the sufficiency is proved

The proof of necessity is immediate from (54) and the following lemma.

Lemma 5.2. Let $F(\eta) = \sum_{A \subset \mathbf{T}_N} f(A)\, \eta^A$ be a local function and satisfies

$$\sum_{x \in \mathbf{T}_N} \tau_x F(\eta) = 0 \tag{40}$$

for all η. Then there exists a local function $g(\eta)$ such that

$$F(\eta) = g(\eta) - \tau_1 g(\eta)$$

Proof. First, we show that the coefficient $f(A)$ satisfies that

$$\sum_{x \in \mathbf{T}_N} f(\tau_x A) = 0 \tag{41}$$

for all $A \subset \mathbf{T}_N$ by induction on the cardinality of A. Let $A = \varnothing$, considering η: $\eta_x = 0$ for all $x \in \mathbf{T}_N$, then we have

$$0 = \sum_{x \in \mathbf{T}_N} f(\tau_x \varnothing)$$

Assume that if $B \subset A$ but $B \neq A$, then $\sum_{x \in \mathbf{T}_N} f(\tau_x B) = 0$. Considering η: $\eta_x = 1$ for all $x \in A$ and $\eta_x = 0$ for all $x \notin A$, then we have

$$0 = \sum_{x \in \mathbf{T}_N} f(\tau_x A) + \sum_{B \subset A,\, B \neq A} \sum_{x \in \mathbf{T}_N} f(\tau_x B)$$

We partition the power set $\mathscr{P}(\mathbf{T}_N)$ into the equivalence classes of congruence. Denote by $\mathscr{T}$ the set of representations, i.e., $\mathscr{T}$ is a family of sets which satisfies that

(i) If $A \in \mathscr{T}$, then $\tau_x A \notin \mathscr{T}$ for $x \neq 0$

(ii) $\{\tau_x A\}_{x \in \mathbf{T}_N,\, A \in \mathscr{T}} = \mathscr{P}(\mathbf{T}_N) \backslash \varnothing$.

By means of $\mathscr{T}$, the function F may be written as

$$F(\eta) = \sum_{A \in \mathscr{T}} \sum_{x \in \mathbf{T}} f(\tau_x A)\, \eta^{\tau_x A}$$

Because $F(\eta)$ is a local function, for each $A \in \mathscr{T}$ $f(\tau_x A)$ vanishes except for a finite number of x. We can choose n and $\{x_i\}_{i=1}^n$ such that $x_i < x_{i+1}$ and $f(\tau_x A) = 0$ if $x \neq x_i$ for all i. Now we decompose

$$\begin{aligned}
&\sum_{x \in \mathbf{T}} f(\tau_x A)\, \eta^{\tau_x A} \\
&\quad = \sum_{i=1}^{n} f(\tau_{x_i})\, \eta^{\tau_{x_i} A} \\
&\quad = f(\tau_{x_1} A)(\eta^{\tau_{x_1} A} - \eta^{\tau_{x_2} A}) + (f(\tau_{x_1} A) + f(\tau_{x_2} A))\, \eta^{\tau_{x_2} A} + \sum_{i=3}^{n} f(\tau_{x_i})\, \eta^{\tau_{x_i} A}
\end{aligned}$$

repeating the same procedure we arrive at

$$\sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} f(\tau_{x_k} A) \right) (\eta^{\tau_{x_i} A} - \eta^{\tau_{x_{i+1}} A}) + \sum_{k=1}^{n} f(\tau_{x_k} A)\, \eta^{\tau_{x_n} A} \tag{42}$$

The second sum is equal to zero according to the equality (41). It is easy to see that the first sum is of the form $g_A(\eta) - \tau_1 g_A(\eta)$. ∎

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